NONEXISTENCE FOR A BOUNDARY VALUE PROBLEM ARISING IN PARABOLIC THEORY

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ABSTRACT

The boundary value problem $c_t = c_{xx} - c_{yy} + q(t,x)c$ with

$$c(t,x,0) = 0, \qquad 2c(t,x,x) = \int_0^x q(t,y)dy$$

was solved by Colton [1] for q analytic in t. The solution may be used for mapping solutions of the heat equation into solutions of $u_t = u_{xx} + q(t,x)u$. Solutions (of the boundary value problem) no longer exist if q is not analytic in t.

1. Introduction and statement of results

Let $q(t,x)$ be a smooth function of x and t. It is well known (Colton [1]) that if q is analytic in t uniformly in x, then there exists a kernel $c = c(t,x,y)$ such that the formula

$$u(t,x) = h(t,x) + \int_0^x c(t,x,y)h(t,y)dy \tag{1.1}$$

transforms solutions h of the standard one-dimensional heat equation

$$h_t = h_{xx} \tag{1.2}$$

with $h(t,0) = 0$ into solutions of

$$u_t = u_{xx} + q(t,x)u \tag{1.3}$$

with $u(t,0) = 0$.

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The kernel c is constructed by Colton as a solution of

$$c_t = c_{xx} - c_{yy} + q(t,x)c \tag{1.4}$$

with

$$c(t,x,0) = 0, \qquad 2c(t,x,x) = \int_0^x q(t,y)dy. \tag{1.5}$$

The problem (1.4), (1.5) is solved by means of a majorant argument similar to the one used in the proof of the Cauchy-Kowalewsky Theorem, and the analyticity of q in t is essential to the proof.

THEOREM. *There exists a C^∞ function $q(t)$ such that the problem* (1.4), (1.5) *has no solutions.*

Seidman [4] applied the transformation (1.1) in order to use the known null boundary controllability result for the heat equation so as to obtain null boundary controllability for (1.3). By null boundary controllability for (1.2) (resp., (1.3)) in the interval $(0,L)$ we mean that for every initial data u_0 defined on $(0,L)$ and every $T > 0$ there exists a function $\varphi(t)$ (the control) defined on $(0,T)$ such that the solution of (1.2) (resp., (1.3)) with the initial conditions $h(0,x) = u_0$ (resp., $u(0,x) = u_0$) for $0 < x < L$ and the boundary conditions $h(t,0) = \varphi(t)$, $h(t,L) = 0$ (resp., $u(t,0) = \varphi(t)$, $u(t,L) = 0$) for $0 < t < T$, satisfies the terminal conditions $h(T,x) = 0$ (resp., $u(T,x) = 0$) for $0 \le x \le L$. The question of null boundary controllability for (1.3) in the case of smooth q is open, as one has null boundary controllability for (1.3) if q is independent of x.

I am very much indebted to Professor T. Seidman for suggesting this problem to me and for a very helpful discussion of it.

2. Proofs

Let $g(t)$ be an infinitely differentiable positive function such that the estimate

$$|g^{(n)}(t)| \le C^{n+2}n^{2n} \tag{2.1}$$

holds for *no* value of C. The existence of such a function $g(t)$ follows, e.g., from a special case of a theorem of Borel (Theorem 1.2.6 in [3]). Set $\mathfrak{Q}(t) = -\ln g(t)$, $q(t) = \mathfrak{Q}'(t)$. We claim that the problem (1.4), (1.5) is not solvable for this choice of q. Assume, on the contrary, that a function $c(t,x,y)$ satisfying (1.4) and (1.5) does exist, so that (1.1) transforms a solution h of the heat equation (1.2) with $h(t,0) = 0$ into a solution u of (1.3). Set $h(t,x) = x$. Then

$$u(t,x) = x + \int_0^x c(t,x,y)\,y\,dy \tag{2.2}$$

satisfies the equation (1.3) with $u(t,0) = 0$ and $u_x(t,0) = 1$. Set now

$$v(t,x) = u(t,x)e^{-\mathcal{Q}(t)}. \tag{2.3}$$

Then

$$v_t = (u_t - qu)e^{-\mathcal{Q}} = (u_{xx} + qu - qu)e^{-\mathcal{Q}} = v_{xx}$$

with $v(t,0) = 0$ and $v_x(t,0) = e^{-\mathcal{Q}(t)} = g(t)$.

By the well-known reflection principle for the heat equation (see, e.g., Theorem 4.6 in [2]), $v(t,x)$ can be defined for $x < 0$ by $v(t,x) = -v(t,-x)$ and continues to be a solution of (1.2). Hence, points on $x = 0$ belong to the interior of the set Ω where v satisfies (1.2).

According to Theorem 11.4.12 in [3] and Example 11.4.15 there, every solution v of the heat equation satisfies the estimates

$$\left| \frac{\partial^{n+m} v(t,x)}{\partial^n t \partial^m x} \right| \le C^{n+m+1} n^{2n} m^m, \qquad n,m = 0,1,\ldots \tag{2.4}$$

in compact subsets of Ω (C depending on v and the set). This contradicts (2.1).

Remark. The problem of null-controllability for (1.3) is still open. While our example shows that the method of [4] is inapplicable for $q(t)$ as above, we can nevertheless steer the solutions of (1.3) (for such a q) to zero. In fact, let $u_0(x) \in L^2(0,L)$ be any initial data and let $T > 0$ be given. There exists a control $\varphi(t)$ [4] such that the solution h of (1.2) with $h(t,0) = \varphi(t)$, $h(t,L) = 0$ satisfies $h(T,x) = 0$. Then the function $u(t,x) = e^{[\mathcal{Q}(t) - \mathcal{Q}(0)]} h(t,x)$ satisfies (1.3), $u(0,x) = u_0$, and $u(T,x) = 0$. Hence $\exp[\mathcal{Q}(t) - \mathcal{Q}(0)]\varphi(t)$ is a control steering u to zero.

References

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3. L. Hörmander, *The Analysis of Linear Partial Differential Operators I, II*, Springer, Berlin, 1983.

4. T. I. Seidman, *Two results on exact boundary control of parabolic equations*, Appl. Math. Optim. **11** (1984), 145–152.